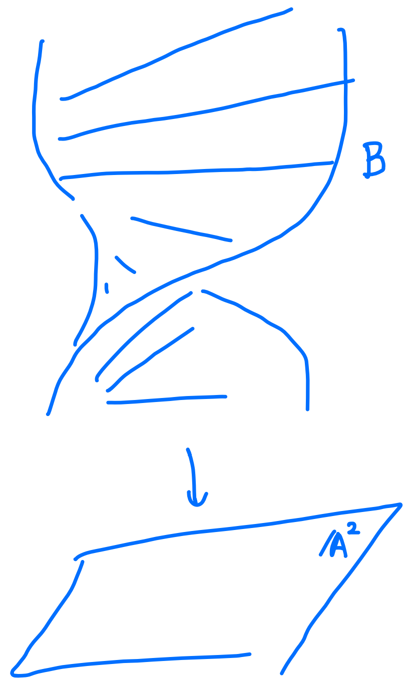
Blowing up of $\mathbb{A}^2$ at $(0,0)$



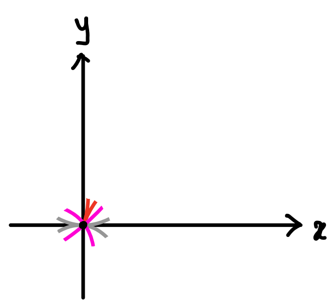

$\forall\ C \in \mathbb{A}^2$

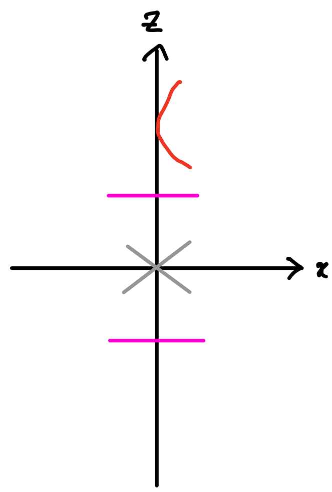
$\Rightarrow\ C' \subset B$

$C' \xrightarrow{\pi} C$ birational equivalent

$C'$ is better than $C$

Example:



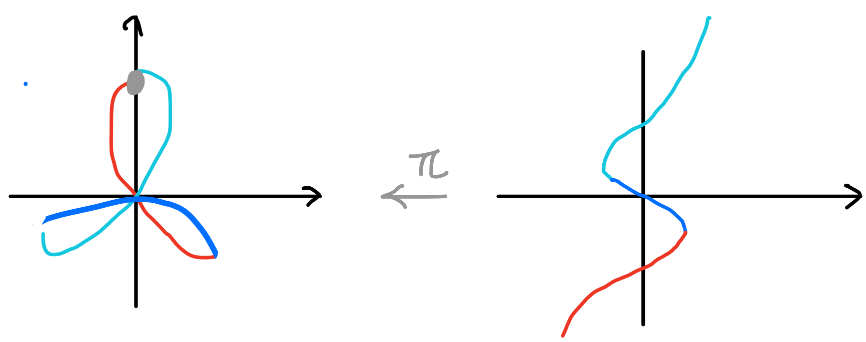




$F = (y-x)\cdot(x+y)\cdot[(y-2x)^2 - x^3]\cdot[y^2 - x^4 + x^5] + x^N$

$F_r = (y-x)\cdot(x+y)(y-2x)^2 y^2$

$F' = (z-1)(z+1)[(z-2)^2 - x][z^2 - x^2 + x^3] + x^{N-6}$

Example.

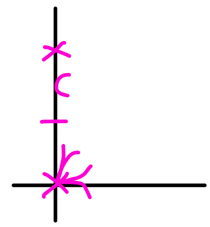


Lem (3). $\exists\ W \overset{\circ}{\hookrightarrow} C$ s.t. $W' = f^{-1}(W) \overset{\circ}{\hookrightarrow} C'$ affine open subvar
$W \ni P$

① $f(W') = W$

② $\Gamma(W')/\Gamma(W)$ = finite with $x^{r-1}\Gamma(W') \subset \Gamma(W)$

Pf: $F = \sum_{i+j\geq r} a_{ij} X^i Y^j \qquad H = \sum_{j \geq r} a_{0j} Y^{j-r} = F(0,Y)/Y^r$



$h = H \bmod I(C) \in \Gamma(C)$.

$H(0,0) = 1 \Rightarrow W = C_h \ni P$ open affine in $C$.

$\Rightarrow W' = f^{-1}(W) = (C')_h$ open affine in $C'$

To prove ①&②, ONTS: $z$ integral over $\Gamma(W)$. i.e.

$$z^r + b_1 z^{r-1} + \cdots + b_r = 0 \qquad (*)$$

for some $b_1, \ldots, b_r \in \Gamma(W)$

$$\left( \begin{array}{l} \left.\begin{array}{l} \text{Since } \Gamma(W') = \Gamma(W)[z] \Rightarrow \Gamma(W') = \sum_{i=0}^{r-1} \Gamma(W)\cdot z^i \\ \qquad\qquad x^{r-1}\cdot z^i \in \Gamma(W) \ \forall i \leq r-1 \end{array}\right\} \Rightarrow \checkmark \\ \text{for any } (x,y)\in W \text{ we can solve } * \text{ find } (x,z)\in W' \,! \end{array} \right)$$

$$F'(x,z) = \sum_{i+j\geq r} a_{ij}\, x^{i+j-r} z^{j} = \sum_{i+j\geq r} a_{ij}\, y^{i+j-r} z^{r-i}$$

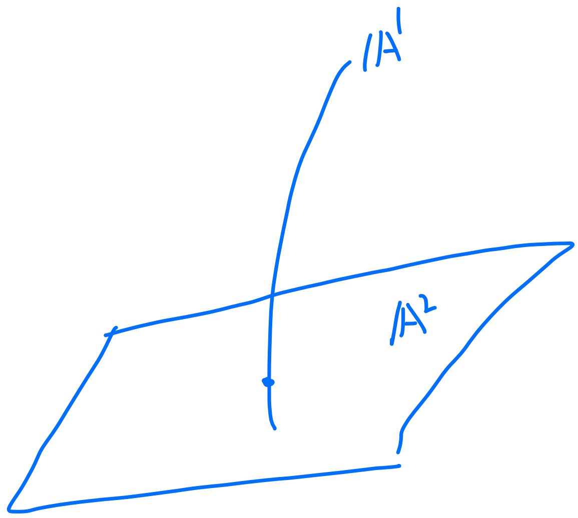
$$= \sum_{i=0}^{r-1} \left( a_{ij}\, y^{i+j-r} \right) \cdot z^{r-i} + \sum_{\substack{i\geq r \\ i+j\geq r}} a_{ij}\, x^{i-r} y^{j}$$

$$\begin{cases} b_i = \dfrac{1}{h} \displaystyle\sum_{j} a_{ij}\, y^{i+j-r} & 1\leq i < r \\ b_r = \dfrac{1}{h} \displaystyle\sum_{\substack{i\geq r \\ j}} a_{ij}\, x^{i-r} y^{j} \end{cases}$$

$$F'(x,z) = 0 \Rightarrow z^r + b_1 z^{r-1} + \cdots + b_r = 0 \quad !$$

$$\begin{tikzcd} \mathbb{A}^2 \arrow[d, "\mathrm{id}"] & B \arrow[l, "\cong"'] \arrow[d, "\pi"'] & \supseteq & C' \arrow[d, "\pi"'] & W' \arrow[l, hook'] \arrow[d, "\pi"] \\ \mathbb{A}^2 & \mathbb{A}^2 \arrow[l, phantom, "="] & \supseteq & C & W \arrow[l, hook'] \end{tikzcd}$$

not surjective & not finite.

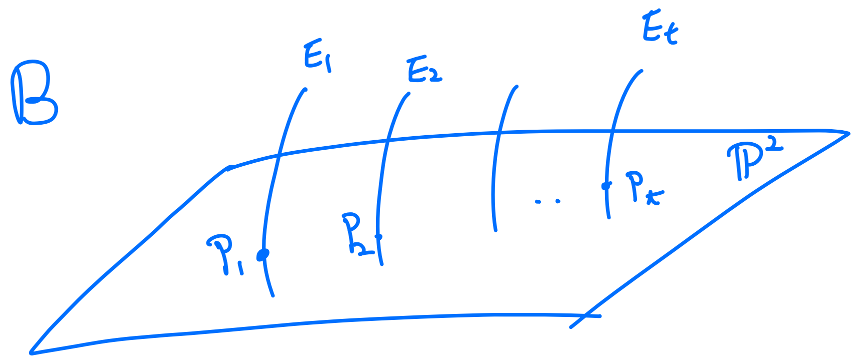



$$f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1 \Rightarrow G(f) \Rightarrow B = \overline{G(f)} = V(y - xz)$$

## §7.3 Blowing up Points in $\mathbb{P}^2$

aim: blow up points $P_1, \cdots, P_t \in \mathbb{P}^2$. i.e. replace each by a projective line

WMA: $P_i = [a_{i_1} : a_{i_2} : 1] \in U_3 \quad \forall i = 1, \cdots, t$.




$U := \mathbb{P}^2 \setminus \{P_1, \cdots, P_t\}$. Define

$$f_i : U \to \mathbb{P}^1 \quad [x_1 : x_2 : x_3] \mapsto [x_1 - a_{i_1} x_3 : x_2 - a_{i_2} x_3] \qquad (*)$$

and

$$f = (f_1, f_2, \cdots, f_t) : U \to \overbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}^{t}$$

$G :=$ graph of $f \subseteq U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$X_1, X_2, X_3$ homogeneous coordinates for $\mathbb{P}^2$

$Y_{i1}, Y_{i2}$ homogeneous coordinates for $i$th $\mathbb{P}^1$

$B :=$ closure of $G$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$$= V\left( \{ Y_{i1}(X_2 - a_{i_2} X_3) - Y_{i2}(X_1 - a_{i_1} X_3) \mid i = 1, \cdots, t \} \right)$$

⑧ $\quad \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$$\begin{tikzcd} B \arrow[r, hook] \arrow[rr, bend right, "\pi"'] & \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[r, "Pr"] & \mathbb{P}^2 \end{tikzcd}$$

i-th $\mathbb{P}^1$

$E_i := \{P_i\} \times \{f_1(P_i)\} \times \cdots \times \mathbb{P}^1 \times \cdots \times \{f_t(P_i)\} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$$E_i \xrightarrow{\sim} \mathbb{P}^1$$

Fact: 1) $G = B \cap (U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = B \setminus \bigcup_{i=1}^{t} E_i$

2) $B \setminus \bigcup_{i=1}^{t} E_i \xrightarrow[\cong]{\pi} U$

study the behavior of $\pi$ around some pt $Q \in E_i$.

Fact: locally $\pi : B \to \mathbb{P}^2$ looks like $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ (in $\S 7.2$)

i.e. $\forall\, Q \in B$ $\exists$ $V' \hookrightarrow B$ & $W' \hookrightarrow \mathbb{A}^2$ s.t.

$$\begin{tikzcd} \mathbb{A}^2 \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & W' \arrow[l, hook'] \arrow[r, "\varphi", "\cong"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\pi"] \\ \mathbb{A}^2 & W \arrow[l, hook'] \arrow[r, "\varphi_3", "\cong"'] & V \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$